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On dispersive properties of discrete 2D Schrödinger and Klein–Gordon equations

A.I. Komech [a,b,1,2], E.A. Kopylova [b,*,2], B.R. Vainberg [c]

[a] Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, Wien A-1090, Austria
[b] Institute of the Information Transmission Problems RAS, B.Karetnii per. 19, Moscow 101447, Russia
[c] Department of Mathematics and Statistics, UNCC, Charlotte, NC 28223, USA

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Abstract

We derive the long-time asymptotics for solutions of the discrete 2D Schrödinger and Klein–Gordon equations.

Keywords: Discrete Schrödinger and Klein–Gordon equations; Lattice; Cauchy problem; Long-time asymptotics

1. Introduction

In this paper we establish the long-time behavior of solutions to the discrete two-dimensional Schrödinger and Klein–Gordon equations. We extend a general strategy introduced by Vainberg [14], Jensen, Kato [6], Murata [10] concerning the wave, Klein–Gordon and Schrödinger equations, to the discrete case. Namely, we establish the smoothness of the resolvent of a stationary problem at the nonsingular points of continuous spectrum, and a generalised 'Puiseux expansion' at the singular points which are critical values of the symbol. Then, the long-time asymptotics can be obtained by means of the (inverse) Fourier–Laplace transform.

* Corresponding author.
E-mail address: ek@vpti.vladimir.ru (E.A. Kopylova).
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We restrict ourselves to the "nonsingular case," in the sense of [10], where the truncated resolvent is bounded at the singular points of the continuous spectrum, i.e. there are no resonances or eigenvalues. This holds generically and allows us to get decay of order $\sim t^{-1}(\log t)^{-2}$ which is desirable for applications to scattering problems.

First, we consider discrete version of the 2D Schrödinger equation

$$\begin{cases} i\dot{\psi}(x,t) = H\psi(x,t) := \big(-\Delta + V(x)\big)\psi(x,t), \\ \psi|_{t=0} = \psi_0, \end{cases} \Bigg| \quad x \in \mathbb{Z}^2,\ t \in \mathbb{R}. \tag{1.1}$$

Here Δ stands for the difference Laplacian in $\mathbb{Z}^2$ defined by

$$\Delta\psi(x) = \sum_{|x-y|=1} \psi(y) - 4\psi(x), \quad x \in \mathbb{Z}^2, \tag{1.2}$$

for functions $\psi : \mathbb{Z}^2 \to \mathbb{C}$.

Definition 1.1. Denote by $\mathcal{V}$ the set of real-valued functions f on the lattice $\mathbb{Z}^2$ with finite support.

Assume that $V \in \mathcal{V}$. If we apply the Fourier–Laplace transform

$$\tilde{\psi}(x,\omega) = \int_0^\infty e^{i\omega t}\psi(x,t)\,dt, \quad \operatorname{Im}\omega > 0, \tag{1.3}$$

to (1.1), then we obtain the stationary equation

$$(H-\omega)\tilde{\psi}(\omega) = -i\psi_0, \quad \operatorname{Im}\omega > 0. \tag{1.4}$$

Note that the integral (1.3) converges, since $\|\psi(\cdot,t)\|_{l^2} = const$ by charge conservation. Hence

$$\tilde{\psi}(\cdot,\omega) = -iR(\omega)\psi_0, \quad \operatorname{Im}\omega > 0, \tag{1.5}$$

where $R(\omega) = (H-\omega)^{-1}$ is the resolvent of the Schrödinger operator H.

We are going to use functional spaces which are discrete versions of the Agmon spaces [1]. These spaces are the weighted Hilbert spaces $l^2_\sigma = l^2_\sigma(\mathbb{Z}^2)$ with the norm

$$\|u\|_{l^2_\sigma} = \big\|\big(1+x^2\big)^{\sigma/2}u\big\|_{l^2}, \quad \sigma \in \mathbb{R}.$$

Let us denote by

$$B(\sigma,\sigma') = \mathcal{L}\big(l^2_\sigma, l^2_{\sigma'}\big), \qquad \mathbf{B}(\sigma,\sigma') = \mathcal{L}\big(l^2_\sigma \oplus l^2_\sigma, l^2_{\sigma'} \oplus l^2_{\sigma'}\big)$$

the spaces of bounded linear operators from l^2_σ to $l^2_{\sigma'}$ and from $l^2_\sigma \oplus l^2_\sigma$ to $l^2_{\sigma'} \oplus l^2_{\sigma'}$, respectively.

Note that the continuous spectrum of the operator H coincides with the interval $[0,8]$, and the kernel of the resolvent has singularities of the logarithmic type at points $\omega_1 = 0$, $\omega_2 = 4$, and $\omega_3 = 8$. The points are critical values of the symbol $4(\sin^2\frac{\theta_1}{2} + \sin^2\frac{\theta_2}{2})$, $(\theta_1\theta_2) \in T^2$ of the difference Laplace operator (1.2), where T^2 is the torus $\mathbb{R}^2/2\pi\mathbb{Z}^2$. The values $\omega_1 = 0$ and $\omega_3 = 8$ correspond to elliptic critical points of the symbol, and $\omega_2 = 4$ corresponds to hyperbolic points on the torus (see [3]).

Our main results are as follows. First, we prove the *limiting absorption principle*

$$R(\omega \pm i\varepsilon) \xrightarrow{B(\sigma,-\sigma)} R(\omega \pm i0) \quad \text{as } \varepsilon \to 0+ \text{ for } \omega \in (0,4) \cup (4,8), \tag{1.6}$$

for $V \in \mathcal{V}$ and $\sigma > 1/2$. Further, we establish asymptotic expansions of the resolvent near the critical points ω_k for potentials $V(x)$ from a generic set W in the space of all compactly supported potentials (see Definition 3.4). For example, in the elliptic case, $k = 1$ and $k = 3$, the expansion reads

$$R(\omega_k + \omega) = R_k^0 + \frac{R_k^1}{a + \log\omega} + \mathcal{O}\big(\omega \log^2 \omega\big), \quad |\omega| \to 0, \tag{1.7}$$

in the norm of $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > 3$. Similar expansion is valid in neighborhoods of hyperbolic points. These expansions immediately imply

$$R(\omega_k + \omega) = R_k^0 + R_k^1 \log^{-1}\omega + \mathcal{O}\big(\log^{-2}\omega\big), \quad |\omega| \to 0. \tag{1.8}$$

Finally, we take the inverse Fourier–Laplace transform of (1.5), applying the asymptotics (1.6) and (1.8). Then we obtain long-time asymptotics

$$\Bigg\| e^{-itH} - \sum_{j=1}^{n} e^{-it\lambda_j} P_j \Bigg\|_{\mathbf{B}(\sigma,-\sigma)} = \mathcal{O}\big(t^{-1}\log^{-2} t\big), \quad t \to \infty, \tag{1.9}$$

which is our main result. Here P_j are the orthogonal projections in l^2 onto the eigenspaces of H, corresponding to the discrete eigenvalues $\lambda_j \in \mathbb{R}$.

For the proof, we first construct the asymptotic expansion near the critical points for the free resolvent. Then we prove (1.7) for $V \in W$, by arguments similar to [6,10,14]. The proof of the decay (1.9) follows by arguments similar to [14, Lemmas 9, 10] and [6, Lemma 10.2].

Remark 1.2.

(i) In Appendix C, we give an alternative approach to derivation of expansion (1.8) in the case when the truncated resolvent is bounded near the critical points. The derivation relies on our results for the free resolvent and methods [3].

(ii) The obtained asymptotics (1.8) imply the boundedness of the truncated resolvent (which is equivalent to the absence of the eigenvalues and resonances, see [10,15]). Hence the boundedness holds for generic potentials.

(iii) Although the expansion (1.8) implies (1.9), a more accurate expansion (1.7) can be useful in other application.

We also obtain similar results for the discrete Klein–Gordon equation

$$\left\{\begin{array}{l} \ddot\psi(x,t) = \big(\Delta - m^2 - V(x)\big)\psi(x,t), \\ \psi|_{t=0} = \psi_0, \ \dot\psi|_{t=0} = \pi_0, \end{array}\right| \quad x \in \mathbb{Z}^2, \ t \in \mathbb{R}. \tag{1.10}$$

Set $\boldsymbol{\Psi}(t) \equiv (\psi(\cdot,t), \dot\psi(\cdot,t))$, $\boldsymbol{\Psi}_0 \equiv (\psi_0, \pi_0)$. Then (1.10) takes the form

$$i\dot{\boldsymbol{\Psi}}(t) = \mathbf{H}\boldsymbol{\Psi}(t), \quad t \in \mathbb{R}; \qquad \boldsymbol{\Psi}(0) = \boldsymbol{\Psi}_0, \tag{1.11}$$

where

$$\mathbf{H} = \begin{pmatrix} 0 & i \\ i(\Delta - m^2 - V) & 0 \end{pmatrix}.$$

The resolvent $\mathbf{R}(\omega) = (\mathbf{H} - \omega)^{-1}$ of the operator $\mathbf{H}$ can be expressed in terms of the resolvent $R(\omega)$, and this expression yields the corresponding properties of $\mathbf{R}(\omega)$. In particular, we derive an asymptotic expansion of type (1.7) for $\mathbf{R}(\omega)$, and also long-time asymptotics of type (1.9) for the solution of (1.11).

Let us comment on previous results in this direction. Eskina [2], and Shaban and Vainberg [12] considered the difference Schrödinger equation in dimension $n \geqslant 1$. They proved the limiting absorption principle for matrix elements of the resolvent and applied it to the Sommerfeld radiation condition. However, [2,12] do not concern the asymptotic expansion of $R(\omega)$ and the long-time asymptotics of type (1.9) in the operator norms.

The asymptotic expansion of the matrix element of the resolvent $R(\omega)$ at the singular points ω_k was obtained by Islami, Vainberg [3]. They used this expansion for obtaining the long-time asymptotics for the solutions of the Cauchy problem for the difference wave equation. The main feature of this paper which differs from [3] in that here, all asymptotic expansions hold in the weighted functional spaces, and not on compacts as in [3]. In fact, the asymptotic expansion of the resolvent (1.7) in the space $B(\sigma, -\sigma)$ was the main technical challenge in this paper. An additional difference is that here we obtain the long-time asymptotics for the Schrödinger and Klein–Gordon equations, as opposed to the wave equation in [3].

For hyperbolic PDEs in $\mathbb{R}^n$ (continuous case), the asymptotic expansion of the resolvent and the long-time asymptotics (1.9) were obtained earlier in [9,13–15], and for the Schrödinger equation in [4–7,10]; see also [11] for an up-to-date review and many references concerning dispersive properties of solutions to the continuous Schrödinger equation in various norms. For the discrete equations, the asymptotic expansion of the resolvent and the long-time asymptotics in the weighted spaces l^2_σ, are obtained for the first time at present paper.

The results of this paper extend the results of [8] from 1D difference equations to 2D difference equations. An exact formula for the resolvent of the stationary equation is used in [8]. An exact formula is missing for 2D problems, and this provides the main difficulty of investigation. Our approach is based on calculation of the asymptotics of oscillatory integrals representing the resolvent.

The paper is organized as follows. In Section 2 we derive the asymptotic expansion of the free resolvent. The limiting absorption principle and the expansion of the perturbed resolvent is proved in Section 3. In Section 4 we prove the long-time asymptotics (1.9). In Section 5 we extend the results to the discrete Klein–Gordon equation.

2. The free resolvent

We start with an investigation of the unperturbed problem for Eq. (1.1) with $V(x) = 0$. The discrete Fourier transform of $u : \mathbb{Z}^2 \to \mathbb{C}$ is defined by the formula

$$\hat u(\theta) = \sum_{x \in \mathbb{Z}^2} u(x) e^{i\theta x}, \quad \theta \in T^2 := \mathbb{R}^2 / 2\pi\mathbb{Z}^2.$$

After taking the Fourier transform, the operator $H_0 = -\Delta$ becomes the operator of multiplication by $\phi(\theta) := 4 - 2\cos\theta_1 - 2\cos\theta_2 = 4\sin^2\frac{\theta_1}{2} + 4\sin^2\frac{\theta_2}{2}$:

$$-\widehat{\Delta u}(\theta) = \phi(\theta)\hat{u}(\theta). \tag{2.1}$$

Thus, the operator H_0 is selfadjoint and its spectrum coincides with the range of the function ϕ, that is $\operatorname{Spec} H_0 = \Sigma := [0, 8]$. Denote by $R_0(\omega) = (H_0 - \omega)^{-1}$ the resolvent of the difference Laplacian. Then the kernel of the resolvent $R_0(\omega)$ reads as

$$R_0(\omega, x-y) = \frac{1}{4\pi^2}\int\limits_{T^2} \frac{e^{-i\theta(x-y)}}{\phi(\theta)-\omega}\,d\theta, \quad \omega \in \mathbb{C}\setminus\Sigma. \tag{2.2}$$

Lemma 2.1. *The free resolvent $R_0(\omega)$ is an analytic function of $\omega \in \mathbb{C}\setminus\Sigma$ with values in $\mathcal{B}(\sigma, \sigma')$ for any $\sigma, \sigma' \in \mathbb{R}$.*

Proof. For a fixed $\omega \in \mathbb{C}\setminus\Sigma$, we have $\phi(\theta) - \omega \neq 0$ for $\theta \in T^2$. Therefore, also $\phi(\theta + i\xi) - \omega \neq 0$ for $\theta \in T^2$, $\xi \in \mathbb{R}^2$ if $\xi \neq 0$ is sufficiently small. Hence, the function $1/(\phi(\theta) - \omega)$ admits an analytic continuation into a complex neighborhood of the torus of type $\{\theta + i\xi\colon \theta \in T^2, \xi \in \mathbb{R}^2, |\xi| < \delta(\omega)\}$ with an $\delta(\omega) > 0$. Therefore, the Paley–Wiener arguments imply that

$$R_0(\omega, x-y) \leqslant C(\delta)e^{-\delta|x-y|}$$

for any $\delta < \delta(\omega)$. Hence, $R_0(\omega)$ is a Hilbert–Schmidt operator in the space $\mathcal{B}(\sigma, \sigma')$. $\square$

2.1. Limiting absorption principle

We are going the traces of the resolvent $R_0(\omega)$ on the continuous spectrum, $R_0(\omega \pm i0)$ with $\omega \in \Sigma$. We can write

$$R_0(\omega \pm i\varepsilon, x-y) = F^{-1}_{\theta\to x-y}\frac{1}{\phi(\theta)-\omega \mp i\varepsilon}, \quad \varepsilon > 0. \tag{2.3}$$

Note that the limiting distribution $\frac{1}{\phi(\theta)-\omega\mp i0}$ is well defined if ω is not a critical value of the function $\phi(\theta)$, i.e. the level line $\phi(\theta) = \omega$ does not contain the critical points with $\nabla\phi(\theta) = 0$. The critical points $\theta = (\theta_1, \theta_2)$ can be easily calculated: $\theta_i = 0, \pm\pi, \ldots$. Therefore, the critical values are 0, 4 and 8. Hence, the limits $R_0(\omega \pm i0, x-y)$ exist, as the distributions of $x-y$, if $\omega \neq 0, 4, 8$. More precisely, the following limiting absorption principle holds in the noncritical part $\Sigma \setminus \{0, 4, 8\}$ of the continuous spectrum.

Proposition 2.2. *For $\sigma > 1/2$ the following limits exist as $\varepsilon \to 0+$:*

$$R_0(\omega \pm i\varepsilon) \xrightarrow{\mathcal{B}(\sigma,-\sigma)} R_0(\omega \pm i0), \quad \omega \in \Sigma\setminus\{0, 4, 8\}. \tag{2.4}$$

Proof. We shall consider the case $\omega \in (0, 4)$ and identify the torus T^2 in (2.2) with the square $[-\pi, \pi]^2$. For $\omega \in (4, 8)$, the proof follows similarly if T^2 is identified with the square $[0, 2\pi]^2$.

Let $\Gamma(\nu)$ be the curve $\{\theta \in T^2\colon\ \phi(\theta) = \nu\}$ for $\nu \in \mathbb{R}$. Denote

$$d = d(\nu) = \min\bigl\{\operatorname{dist}\bigl(\Gamma(\nu), 0\bigr), \operatorname{dist}\bigl(\Gamma(\omega), \Gamma(4)\bigr)\bigr\},$$

and represent $R_0(\omega + i\varepsilon, z)$ as the sum

$$\begin{aligned} R_0(\omega + i\varepsilon, z) &= \frac{1}{4\pi^2} \int_{T^2} \frac{(1-\chi(\theta))e^{-i\theta z}}{\phi(\theta) - \omega - i\varepsilon}\, d\theta + \frac{1}{4\pi^2} \int_{T^2} \frac{\chi(\theta)e^{-i\theta z}}{\phi(\theta) - \omega - i\varepsilon}\, d\theta \\ &= P_1(\omega + i\varepsilon, z) + P_2(\omega + i\varepsilon, z), \end{aligned}$$

where χ is a smooth function on T^2 such that

$$\chi(\theta) = \begin{cases} 0, & |\phi(\theta) - \omega| > d/2, \\ 1, & |\phi(\theta) - \omega| \leqslant d/4. \end{cases}$$

Obviously,

$$\begin{aligned} & P_2(\omega + i\varepsilon, z) \to P_2(\omega + i0, z), \quad \varepsilon \to 0; \\ & \bigl|P_2(\omega + i\varepsilon, z)\bigr| \leqslant \frac{C(\omega, N)}{(|z|^N + 1)}, \quad 0 < \varepsilon \leqslant 1. \end{aligned} \tag{2.5}$$

Let us prove that

$$\begin{aligned} & P_1(\omega + i\varepsilon, z) \to P_1(\omega + i0, z), \quad \varepsilon \to 0; \\ & \bigl|P_1(\omega + i\varepsilon, z)\bigr| \leqslant \frac{C(\omega)}{1 + \sqrt{|z|}}, \quad 0 < \varepsilon \leqslant \delta, \end{aligned} \tag{2.6}$$

for sufficiently small $\delta > 0$. We represent P_1 in the form

$$P_1(\omega \pm i\varepsilon, z) = \frac{1}{4\pi^2} \int\limits_{|\nu - \omega| < d/2} \frac{f(\nu, z)\, d\nu}{\nu - \omega + i\varepsilon}, \quad f(\nu, z) = \int\limits_{\Gamma(\nu)} \frac{\chi(\theta)e^{-i\theta z}\, ds}{|\nabla\phi|}, \tag{2.7}$$

where ds is the length on $\Gamma(\nu)$. The critical points of the phase function $\Phi(\theta) = \theta z$ on $\Gamma(\nu)$ are the points where $\nabla\phi(\theta)$ is proportional to $\alpha = z/|z|$ if $z \neq 0$. Denote the critical points by $\theta_j = \theta_j(\nu, \alpha)$, $j = 1, 2$. Hence, the stationary phase method applied to the integral over $\Gamma(\nu)$ leads to the following result (see [15, Chapter 1, Theorem 9]):

$$f(\nu, z) = \sum_{j=1}^{2} |z|^{-1/2} a_j(\nu, \alpha) e^{i\mu_j(\nu,\alpha)|z|} + r(\nu, z). \tag{2.8}$$

Here a_j, μ_j are smooth functions which are analytic in ν in a neighborhood of the point $\nu = \omega$, $\mu_j = \langle\theta_j, \alpha\rangle$, and

$$\bigl|r(\nu, z)\bigr| \leqslant C/|z|^{3/2}, \qquad \bigl|\partial_\nu r(\nu, z)\bigr| \leqslant C/|z|^{1/2}, \quad |z| \geqslant 1.$$

Let us note that

$$\partial_\nu \mu_j(\nu, \alpha) = \langle \partial_\nu \theta_j, \alpha \rangle \neq 0 \tag{2.9}$$

(one can also find this relation in [15, Chapter VII]). In fact, $\phi(\theta_j) = \nu$. After differentiation in ν we get $\langle \nabla \phi(\theta_j), \partial_\nu \theta_j \rangle = 1$ which implies (2.9), since the vector $\nabla \phi(\theta_j)$ is parallel to α.

Now we split P_1 in three terms $P_1 = P_{11} + P_{12} + P_{13}$ which appear after the function f in (2.7) is replaced by the main terms and the remainder in (2.8). Obviously, (2.6) holds for P_{13}. Let us show that it also holds for P_{11} and P_{12}. The proofs in both cases are similar. So we shall consider the estimates of P_{11}. Assume that $\partial_\nu \mu_1 > 0$. Consider a small $\delta > 0$ and denote by $\gamma_{2\delta}$ contour in the complex ν-plane which consists of two segments $2\delta \leqslant |\nu - \omega| \leqslant d/2$, $\nu \in \mathbb{R}$, and the half circle $|\nu| = 2\delta$, $\operatorname{Im} \nu > 0$. Then by the Cauchy theorem

$$\left| \int\limits_{|\nu-\omega|<d/2} \frac{a_1 e^{i\mu_1 |z|}\, d\nu}{\nu - \omega + i\varepsilon} \right| = \left| \int\limits_{\gamma_{2\delta}} \frac{a_1 e^{i\mu_1 |z|}\, d\nu}{\nu - \omega + i\varepsilon} \right| \leqslant C, \quad 0 < \varepsilon < \delta, \tag{2.10}$$

since $\operatorname{Re} i\mu_1(\nu, \alpha) \leqslant 0$ for $\nu \in \gamma_{2\delta}$ if $\delta > 0$ is small enough. The latter follows from the Lagrange formula applied to $\operatorname{Re} i\mu_j$.

The estimate (2.10) implies (2.6) for P_{11} when $\partial_\nu \mu_1 > 0$. For $\partial_\nu \mu_1 < 0$, we apply the same arguments with $\gamma_{2\delta}$ replaced by the complex conjugate contour and with an extra term in the middle part of the relations (2.10). This term is the residue at point $\nu = \omega - i\varepsilon$. Hence, (2.6) is proved for P_{11}, and therefore it holds for P_1.

The limits and uniform bounds (2.5) and (2.6) imply similar features for the matrix elements $R_0(\omega, z)$ of the resolvent:

$$\begin{aligned} &R_0(\omega \pm i\varepsilon, z) \to R_0(\omega \pm i0, z), \quad \varepsilon \to 0+; \\ &\big|R_0(\omega \pm i\varepsilon, z)\big| \leqslant \frac{C(\omega)}{1 + \sqrt{|z|}}, \quad 0 < \varepsilon \leqslant \varepsilon(\omega). \end{aligned} \tag{2.11}$$

Finally, (2.11) implies that

$$\sum_{x,y \in \mathbb{Z}^2} \big(1 + |x|^2\big)^{-\sigma} \big|R_0(\omega \pm i\varepsilon, x - y) - R_0(\omega \pm i0, x - y)\big|^2 \big(1 + |y|^2\big)^{-\sigma} \to 0, \quad \varepsilon \to 0+,$$

by the Lebesgue dominated convergence theorem. Hence, the Hilbert–Schmidt norm of the difference $R_0(\omega \pm i\varepsilon) - R_0(\omega \pm i0)$ converges to zero that implies (2.4) for $\omega \in (0, 4)$. □

Remark 2.3. Differentiating (2.2) in ω, we obtain similarly that $\partial_\omega^k R_0(\omega \pm i0) \in \mathcal{B}(\sigma, -\sigma)$ with $\sigma > 1/2 + k$ for $k \in \mathbb{N}$ and $\omega \in (0, 4) \cup (4, 8)$.

Further, we need more information on the behavior of $R_0(\mu)$ near the points ω_k. We consider separately the "elliptic" singular points $\omega_1 = 0$, $\omega_3 = 8$ and the "hyperbolic" singular points $\omega_2 = 4$.

2.2. Asymptotic expansion near elliptic points

Here we construct the expansion of the free resolvent $R_0(\omega)$ near the elliptic singular points $\omega_1 = 0$ and $\omega_3 = 8$. First we consider $\omega_1 = 0$ and expand $R_0(\omega)$ for small complex ω for which $0 < \arg\omega < 2\pi$.

Proposition 2.4. *For any $N \geqslant 0$ the following expansion holds*:

$$R_0(\omega) = \sum_{k=0}^{N} A_k \omega^k \log\omega + \sum_{k=0}^{N} B_k \omega^k + \mathcal{O}\bigl(\omega^{N+1}\log\omega\bigr),$$
$$|\omega| \to 0,\ \arg\omega \in (0, 2\pi) \tag{2.12}$$

in the norm of $\mathcal{B}(\sigma,-\sigma)$ with $\sigma > 2N+3$. Here $A_k, B_k \in \mathcal{B}(\sigma,-\sigma)$ with $\sigma > 2k+1$ are the operators with kernels $A_k(x-y)$, $B_k(x-y)$, respectively, and

$$A_0(x-y) = -\frac{1}{4\pi}, \quad x, y \in \mathbb{Z}^2. \tag{2.13}$$

Proof. The resolvent $R_0(\omega)$ is represented by the integral (2.2). Let us fix $0 < \delta < 1$ and consider $0 < |\omega| < \delta/2$. We identify T^2 with the square $[-\pi,\pi]^2$, and split $R_0(\omega,z)$, $z = x - y$, into the sum

$$R_0(\omega,z) = \frac{1}{4\pi^2} \int\limits_{|\phi(\theta)|<\delta} \frac{e^{-i\theta z}}{\phi(\theta)-\omega}\, d\theta + r(\omega,z), \tag{2.14}$$

where $r(\omega,z)$ is the integral over $\{|\phi(\theta)| > \delta\}$. The remainder $r(\omega,z)$ is analytic in ω for $|\omega| \leqslant \delta/2$, and

$$\bigl|\partial_\omega^j r(\omega,z)\bigr| \leqslant C_j, \quad |\omega| \leqslant \delta/2,\ z \in \mathbb{Z}^2. \tag{2.15}$$

Now we change the variables:

$$\alpha_i = 2\sin\frac{\theta_i}{2}, \qquad d\alpha_i = \cos\frac{\theta_i}{2}\, d\theta_i, \quad i = 1, 2.$$

Let us note that the transform $(\theta_1,\theta_2) \mapsto (\alpha_1,\alpha_2)$ is a diffeomorphism of $D = \{|\phi(\theta)| < \delta\}$ since $\cos\frac{\theta_i}{2} \neq 0$ for $(\theta_1,\theta_2) \in D$. Using the symmetry of the domain D in θ_i, $i = 1, 2$, we obtain from (2.14)

$$R_0(\omega,z) = \frac{1}{4\pi^2} \int\limits_{|\alpha_1^2+\alpha_2^2-\omega|<\delta/2} \frac{\cos(z_1 g(\alpha_1))\cos(z_2 g(\alpha_2))}{\alpha_1^2+\alpha_2^2-\omega\mp i\varepsilon} J\bigl(\alpha_1^2\bigr) J\bigl(\alpha_2^2\bigr)\, d\alpha_1\, d\alpha_2 + r(\omega,z). \tag{2.16}$$

Here $g(\alpha) = 2\arcsin\alpha/2$ is a smooth, odd function, and $J(\alpha^2) = \frac{1}{\sqrt{1-\alpha^2/4}}$ is a smooth even function.

Now we change variables further, $\alpha_1 = \sqrt{\rho}\cos\psi$, $\alpha_2 = \sqrt{\rho}\sin\psi$, where $\rho = \alpha_1^2 + \alpha_2^2 = \phi(\theta)$, and denote the integral with respect to ψ by f:

$$f(\rho, z) = \frac{1}{8\pi^2}\int_0^{2\pi} \cos\bigl(z_1 g(\sqrt{\rho}\cos\psi)\bigr)\cos\bigl(z_2 g(\sqrt{\rho}\sin\psi)\bigr) J\bigl(\rho\cos^2\psi\bigr) J\bigl(\rho\sin^2\psi\bigr)\, d\psi. \tag{2.17}$$

Then (2.16) becomes

$$R_0(\omega, z) = \int_0^{\delta} \frac{f(\rho, z)\, d\rho}{\rho - \omega} + r(\omega, z). \tag{2.18}$$

It remains to prove asymptotics of type (2.12) for the integral in (2.18). It is easy to show that

$$\bigl|\partial_\rho^k f(\rho, z)\bigr| \leqslant C_k\bigl(1 + |z|^{2k}\bigr), \quad k = 0, 1, 2, \ldots,\ 0 \leqslant \rho \leqslant \delta. \tag{2.19}$$

For any $N \geqslant 0$ let us expand $f(\rho, z)$ in finite Taylor series in ρ:

$$f(\rho, z) = f_0(z) + f_1(z)\rho + \cdots + f_N(z)\rho^N + F_N(\rho, z)\rho^N, \quad f_0(z) = \frac{1}{4\pi}, \tag{2.20}$$

where $f_k(z)$ are polynomial in z of order $2k$, and

$$\bigl|F_N(\rho, z)\bigr| \leqslant C|z|^{2N}, \qquad \bigl|\partial_\rho F_N(\rho, z)\bigr| \leqslant C|z|^{2N+2}, \quad \text{for } 0 \leqslant \rho \leqslant \delta. \tag{2.21}$$

Substituting (2.20) into (2.18), we obtain for the terms containing $f_k(z)$, $k = 0, 1, \ldots, N$,

$$\begin{aligned}
\int_0^{\delta} \frac{f_k(z)\rho^k\, d\rho}{\rho - \omega} &= f_k(z)\int_0^{\delta}\left(\rho^{k-1} + \omega\rho^{k-2} + \cdots + \omega^{k-1} + \frac{\omega^k}{\rho - \omega}\right) d\rho \\
&= f_k(z)\left(\sum_{j=0}^{k-1} a_j\omega^j + \omega^k\bigl(\log(\delta - \omega) - \log(-\omega)\bigr)\right) \\
&= f_k(z)\left(\sum_{j=0}^{N} a_j\omega^j - \omega^k\log(-\omega) + \hat{a}_N(\omega)\omega^N\right), \quad 0 < |\omega| < \delta/2,
\end{aligned} \tag{2.22}$$

where $|\hat{a}_N(\omega)| \leqslant C$. Further, similar to (2.22) we have

$$\begin{aligned}
\int_0^{\delta} \frac{F_N(\rho, z)\rho^N\, d\rho}{\rho - \omega} &= \int_0^{\delta} F_N(\rho, z)\left(\rho^{N-1} + \omega\rho^{N-2} + \cdots + \omega^{N-1} + \frac{\omega^N}{\rho - \omega}\right) d\rho \\
&= \sum_{j=0}^{N-1} b_{N,j}(z)\omega^j + \omega^N\int_0^{\delta}\frac{F_N(\rho, z)\, d\rho}{\rho - \omega},
\end{aligned} \tag{2.23}$$

where $b_{N,j}(z) \leqslant C|z|^{2N}$. We estimate the last integral using the following lemma.

Lemma 2.5. *For* $0 < |\omega| < \delta/2$, $\arg\omega \in (0, 2\pi)$ *the following bound holds*:

$$\left| \int_0^\delta \frac{F_N(\rho, z)\,d\rho}{\rho - \omega} \right| \leqslant C|z|^{2N}\big(\ln|z| + \big|\ln|\omega|\big|\big), \quad |z| > 1. \tag{2.24}$$

We prove this lemma in Appendix A. Therefore (2.15)–(2.24) imply

$$R_0(\omega, z) = \sum_{k=0}^N A_k(z)\omega^k \log(-\omega) + \sum_{k=0}^N B_k(z)\omega^k + \omega^N \widehat{A}_N(\omega, z), \quad |\omega| \to 0,$$

where $|\widehat{A}_N(\omega, z)| \leqslant C|z|^{2N}(\ln|z| + |\ln|\omega||)$, and $A_k(z) = \mathcal{O}(|z|^{2k})$, while $B_k(z) = \mathcal{O}(|z|^{2N})$ for $0 \leqslant k \leqslant N$. Hence, $B_k(z) = \mathcal{O}(|z|^{2k})$ since $B_k(z)$ does not depend on N. □

Finally, in the case of $\omega_3 = 8$ we obtain similarly, the expansion of $R_0(8 - \omega)$ for small $|\omega| > 0$, $\arg\omega \in (-\pi, \pi)$, with $A_0(z) = \frac{(-1)^{z_1+z_2}}{4\pi}$.

2.3. Asymptotic expansion near hyperbolic points

Here we study the asymptotics at the hyperbolic singular point $\omega_2 = 4$. The main contribution to the integral (2.2) is given by the corresponding critical points $(0, \pi)$ and $(\pi, 0)$ of hyperbolic type.

Proposition 2.6. *For any* $N \geqslant 0$ *the following expansion holds*

$$R_0(4 + \omega) = \sum_{k=0}^N D_k\omega^k \log\omega + \sum_{k=0}^N E_k\omega^k + \mathcal{O}\big(\omega^{N+1}\log\omega\big),$$
$$|\omega| \to 0,\ \operatorname{Im}\omega > 0, \tag{2.25}$$

in the norm of $\mathcal{B}(\sigma, -\sigma)$ *with* $\sigma > 2N + 3$. *Here* $D_k, E_k \in \mathcal{B}(\sigma, -\sigma)$, *with* $\sigma > 2k + 1$, *are operators with kernels* $D_k(x - y)$, $E_k(x - y)$, *respectively, and* $D_0(z) = \frac{-i}{4\pi}((-1)^{z_1} + (-1)^{z_2})$.

For $\operatorname{Im}\omega < 0$ *a similar expansion holds.*

Proof. For $\omega = \omega_2 = 4$ the denominator of the integral (2.2) vanishes along the curve $\phi(\theta) = 4$. We will study the main contribution of points $(0, \pi)$ and $(\pi, 0)$ of the curve which are critical points of $\phi(\theta)$. The contribution of other points on the curve is contained in second sum in the right-hand side of (2.25) and can be proved by methods of Section 2.1. For example, consider the integral over a neighborhood of the point $(\pi, 0)$. Let us introduce a smooth cutoff function $\zeta(\theta)$, which will be specified below, such that $\zeta(\theta) = 1$ in a neighborhood of the point $(\pi, 0)$, and define

$$
\begin{aligned}
Q(\omega, z) &= \frac{1}{4\pi^2} \int \frac{e^{-i(z_1\theta_1+z_2\theta_2)}\zeta(\theta)\,d\theta_1\,d\theta_2}{\phi(\theta)-4-\omega} \\
&= \frac{1}{4\pi^2} \int \frac{e^{-i(z_1\theta_1+z_2\theta_2)}\zeta(\theta)\,d\theta_1\,d\theta_2}{4\sin^2\frac{\theta_2}{2}-4\cos^2\frac{\theta_1}{2}-\omega} \\
&= \frac{e^{-iz_1\pi}}{4\pi^2} \int \frac{e^{-i(z_1\theta_1'+z_2\theta_2)}\zeta_1(\theta')\,d\theta_1'\,d\theta_2}{4\sin^2\frac{\theta_2}{2}-4\sin^2\frac{\theta_1'}{2}-\omega}, \quad \operatorname{Im}\omega > 0,
\end{aligned}
$$

where $\theta_1' = \theta_1 - \pi$, $\theta' = (\theta_1', \theta_2)$, and $\zeta_1(\theta') = \zeta(\theta)$. We can assume that $\zeta_1(\theta')$ is symmetric in θ_1' and θ_2. Then, the exponent $e^{-i(z_1\theta_1'+z_2\theta_2)}$ can be substituted by its even part as above, so that

$$
Q(\omega, z) = \frac{e^{-iz_1\pi}}{\pi^2} \int_0^\infty\int_0^\infty \frac{\cos(z_1\theta_1')\cos(z_2\theta_2)\zeta_1(\theta')\,d\theta_1'\,d\theta_2}{4\sin^2\frac{\theta_2}{2}-4\sin^2\frac{\theta_1'}{2}-\omega} = \frac{e^{-iz_1\pi}}{\pi^2} Q_1(\omega, z). \tag{2.26}
$$

It remains to prove the expansion of type (2.25) for Q_1. First we change the variables $s_1 = 2\sin\frac{\theta_2}{2}$ and $s_2 = 2\sin\frac{\theta_1'}{2}$. Now we specified the cutoff function such that $\zeta_1(\theta') = \zeta_2(|s|^2)$, where ζ_2 is a smooth function. Then

$$
Q_1(\omega, z) = \int_0^\infty\int_0^\infty \frac{F(z, s_1^2, s_2^2)\zeta_2(|s|^2)\,ds_1\,ds_2}{s_1^2 - s_2^2 - \omega}, \tag{2.27}
$$

where

$$
F\big(z, s_1^2, s_2^2\big) = \cos\big(z_1 g(s_1)\big)\cos\big(z_2 g(s_2)\big) J\big(s_1^2\big) J\big(s_2^2\big).
$$

Further, we introduce hyperbolic coordinates by

$$
\rho_1 = s_1^2 - s_2^2 = R^2\cos 2\psi, \qquad \rho_2 = 2s_1s_2 = R^2\sin 2\psi, \tag{2.28}
$$

where R and ψ are polar coordinates of (s_1, s_2). Then $|\rho|^2 = \rho_1^2 + \rho_2^2 = R^4$, so $|\rho| = R^2$, and

$$
s_1^2 = \frac{|\rho| + \rho_1}{2}, \qquad s_2^2 = \frac{|\rho| - \rho_1}{2}. \tag{2.29}
$$

Further, $d\rho_1\,d\rho_2 = 4|\rho|\,ds_1\,ds_2$, hence (2.27) becomes

$$
Q_1(\omega, z) = \int_0^\infty \left(\int_{\mathbb{R}} \frac{h(|\rho|, \rho_1, z)}{(\rho_1 - \omega)|\rho|}\,d\rho_1 \right) d\rho_2, \tag{2.30}
$$

where $h(|\rho|, \rho_1, z) = F(z^2, \frac{|\rho|+\rho_1}{2}, \frac{|\rho|-\rho_1}{2})\zeta_2(|\rho|)/4$. Now, we can further select the cutoff function in such a way that we can choose it so that

$$
\operatorname{supp}\zeta_2\big(|\rho|\big) \cap \big\{\rho \in \mathbb{R}^2\colon\ \rho_2 \geqslant 0\big\} \subset \Pi = \big\{(\rho_1, \rho_2)\colon\ -\delta \leqslant \rho_1 \leqslant \delta,\ 0 \leqslant \rho_2 \leqslant \delta\big\}
$$

with some fixed $0<\delta<1$. We will consider $|\omega|\leqslant\delta/2$. Denote $r=r(\rho):=|\rho|$. Expanding the function $h(r,\rho_1,z)$ in a finite Taylor series in ρ_1 we obtain

$$\begin{aligned}
&h(r,\rho_1,z)=h_0(r,z)+h_1(r,z)\rho_1+\cdots+h_N(r,z)\rho_1^N+H_N(r,\rho_1,z)\rho_1^N,\\
&(\rho_1,\rho_2)\in[-\delta,\delta]\times[0,\delta],\ r=|\rho|,
\end{aligned}\tag{2.31}$$

where $h_k(r,z)$ are polynomials in z of order $2k$, and

$$\begin{aligned}
&\bigl|H_N(r,\rho_1,z)\bigr|\leqslant C|z|^{2N},\qquad \bigl|\partial_{\rho_1}H_N(r,\rho_1,z)\bigr|\leqslant C|z|^{2N+2},\\
&(\rho_1,\rho_2)\in[-\delta,\delta]\times[0,\delta],\ r=|\rho|.
\end{aligned}\tag{2.32}$$

Step (i). Let us consider the contribution to integral (2.30) from the terms containing $h_k(r,z)$, $k=0,1,\dots,N$:

$$\begin{aligned}
\int\limits_{\Pi}\frac{h_k(r,z)\rho_1^k\,d\rho_1\,d\rho_2}{(\rho_1-\omega)r}&=\int\limits_{\Pi}\frac{h_k(r,z)}{r}\left(\rho_1^{k-1}+\omega\rho_1^{k-2}+\cdots+\omega^{k-1}+\frac{\omega^k}{\rho_1-\omega}\right)d\rho_1\,d\rho_2\\
&=\sum_{j=0}^{k-1}a_{k,j}(z)\omega^j+\omega^k\int\limits_{\Pi}\frac{h_k(r,z)}{(\rho_1-\omega)r}\,d\rho_1\,d\rho_2\\
&=\sum_{j=0}^{k-1}a_{k,j}(z)\omega^j+\omega^k\int\limits_0^{\delta}h_k(r,z)\,dr\int\limits_0^{\pi}\frac{d\psi}{r\cos\psi-\omega},
\end{aligned}\tag{2.33}$$

where $a_{k,j}(z)$ are polynomial in z of order $2k$. We change the variable $\tau=\tan(\psi/2)$ to obtain

$$\int\limits_0^{\pi}\frac{d\psi}{r\cos\psi-\omega}=\int\limits_0^{\infty}\frac{d\tau}{-(r+\omega)\tau^2+(r-\omega)}=\frac{\pi i}{\sqrt{r^2-\omega^2}},\quad \operatorname{Im}\omega>0.\tag{2.34}$$

Therefore, (2.33) implies

$$\int\limits_{\Pi}\frac{h_k(r,z)\rho_1^k\,d\rho_1\,d\rho_2}{(\rho_1-\omega)r}=\sum_{j=0}^{k-1}a_j(z)\omega^j+\pi i\omega^k\int\limits_0^{\delta}\frac{h_k(r,z)\,dr}{\sqrt{r^2-\omega^2}}.\tag{2.35}$$

Further, let us expand $h_k(r,z)$ in finite Taylor series in r:

$$h_k(r,z)=h_{k,0}(z)+h_{k,1}(z)r+\cdots+h_{k,N-k}(z)r^{N-k}+H_{k,N-k}(r,z)r^{N-k},\tag{2.36}$$

where $h_{0,0}(z) = 1$, $h_{k,j}(z)$ are polynomial in z of order $2(k+j)$, and $|H_{k,N-k}(r,z)| \leqslant C|z|^{2N}$, $0 \leqslant r \leqslant \delta$. Substituting (2.36) into (2.35), we obtain for the terms containing $h_{k,j}(z)$, with $j = 0, 1, \ldots, N-k$

$$\int_0^\delta \frac{h_{k,j}(z) r^j \, dr}{\sqrt{r^2 - \omega^2}} = h_{k,j}(z) \left(s_j \omega^j \log \omega + \sum_{l=0}^{N-k} \beta_l \omega^l + \hat{\beta}_{N-k}(\omega) \omega^{N-k} \right), \tag{2.37}$$

where $|\hat{\beta}_{N-k}(\omega)| \leqslant B_{N-k}$, $0 < |\omega| < \delta/2$.

It remains to estimate the contribution to the integral in (2.35) from the remainders $H_{k,N-k}(r,z) r^{N-k}$ for $0 < |\omega| < \delta/2$:

$$\int_0^\delta \frac{H_{k,N-k}(r,z) r^{N-k} \, dr}{\sqrt{r^2 - \omega^2}} = \int_0^{2|\omega|} + \int_{2|\omega|}^\delta = I_1 + I_2. \tag{2.38}$$

In the first summand in (2.38) we change the variable $r = |\omega|\tau$ to obtain

$$|I_1| = \left| \int_0^2 \frac{H_{k,N-k}(|\omega|\tau, z) |\omega|^{N-k} \tau^{N-k} \, d\tau}{\sqrt{\tau^2 - \omega^2/|\omega|^2}} \right| \leqslant C|z|^{2N} |\omega|^{N-k}. \tag{2.39}$$

It remains to consider the second summand in (2.38). For odd values of $N-k$ we obtain:

$$\begin{aligned}
I_2 &= \int_{2|\omega|}^\delta H_{k,N-k}(r,z) r^{N-k-1} \left(1 + d_2 \frac{\omega^2}{r^2} + \cdots + d_{N-k-1} \frac{\omega^{N-k-1}}{r^{N-k-1}} + \hat{d}_{N-k+1}(\omega/r) \frac{\omega^{N-k+1}}{r^{N-k+1}} \right) dr \\
&= \int_{2|\omega|}^\delta H_{k,N-k}(r,z) \bigl(r^{N-k-1} + d_2 \omega^2 r^{N-k-3} + \cdots + d_{N-k-1} \omega^{N-k-1} \bigr) \, dr + \tilde{u}_{N-k}(\omega, z) \\
&= \int_0^\delta H_{k,N-k}(r,z) \bigl(r^{N-k-1} + d_2 \omega^2 r^{N-k-3} + \cdots + d_{N-k-1} \omega^{N-k-1} \bigr) \, dr + \hat{u}_{N-k}(\omega, z) \\
&= \sum_{j=0}^{N-k-1} u_j(z) \omega^j + \hat{u}_{N-k}(\omega, z),
\end{aligned} \tag{2.40}$$

where $|u_j(z)| \leqslant C|z|^{2N}|$; $|\tilde{u}_{N-k}(\omega, z)|$, $|\hat{u}_{N-k}(\omega, z)| \leqslant C|z|^{2N} |\omega|^{N-k}$. Similarly, for even values of $N-k$ we obtain

$$I_2 = \sum_{j=0}^{N-k-2} v_j(z) \omega^j + \hat{v}_{N-k}(\omega, z), \quad \bigl| v_j(z) \bigr| \leqslant C|z|^{2N}, \ \bigl| \hat{v}_{N-k}(\omega, z) \bigr| \leqslant C|z|^{2N} |\omega|^{N-k} |\log \omega|.$$

Step (ii). Let us consider the contribution to integral (2.30) from the remainder $H_N(r,\rho_1,z)\rho_1^N$:

$$\int_{\Pi} \frac{H_N(r,\rho_1,z)\rho_1^N\,d\rho_1\,d\rho_2}{(\rho_1-\omega)r}$$
$$= \int_{\Pi} \frac{H_N(r,\rho_1,z)}{r}\left(\rho_1^{N-1} + \omega\rho_1^{N-2} + \cdots + \omega^{N-1} + \frac{\omega^N}{\rho_1-\omega}\right) d\rho_1\,d\rho_2$$
$$= \sum_{j=0}^{N-1} w_j(z)\omega^j + \omega^N \int_{\Pi} \frac{H_N(r,\rho_1,z)\,d\rho_1\,d\rho_2}{(\rho_1-\omega)r}, \tag{2.41}$$

where $|w_j(z)| \leqslant C|z|^{2N}$. The integral in the right-hand side of (2.41) is estimated by the following lemma.

Lemma 2.7. *For* $0 < |\omega| < \delta/2$, $\operatorname{Im}\omega > 0$ *the following bound holds*:

$$\left|\int_{\Pi} \frac{H_N(r,\rho_1,z)\,d\rho_1\,d\rho_2}{(\rho_1-\omega)r}\right| \leqslant C|z|^{2N}\left(\ln^2|z| + \bigl|\ln|\omega|\bigr|\right), \quad |z| > 1. \tag{2.42}$$

We prove this lemma in Appendix B.

Step (iii). Finally, we have proved the expansion

$$Q_1(\omega,z) = \sum_{k=0}^{N} D_k(z)\omega^k \log\omega + \sum_{k=0}^{N} E_k(z)\omega^k + \widehat{E}_N(\omega,z), \quad |\omega| \to 0,$$

where $|\widehat{E}_N(\omega,z)| \leqslant C|z|^{2N}(\ln^2|z| + |\ln|\omega||)|\omega|^N$, and $D_k(z) = \mathcal{O}(|z|^{2k})$, while $E_k(z) = \mathcal{O}(|z|^{2N})$ for $0 \leqslant k \leqslant N$. Hence, $E_k(z) = \mathcal{O}(|z|^{2k})$ since $E_k(z)$ does not depend on N. $\square$

Remark 2.8. Expansions (2.12) and (2.25) can be differentiated $2N+2$ times in ω. For example,

$$\partial_\omega^k R_0(\omega) = \partial_\omega^k\left(\sum_{k=0}^{N} A_k\omega^k \log\omega + \sum_{k=0}^{N} B_k\omega^k\right) + \mathcal{O}\bigl(\omega^{N+1-k}\log\omega\bigr),$$
$$|\omega| \to 0,\ \arg\omega \in (0,2\pi),\ 1 \leqslant k \leqslant 2N+2,$$

in the norm of $\mathcal{B}(\sigma,-\sigma)$ with $\sigma > 2N+3$.

3. Perturbed resolvent

3.1. The limiting absorbtion principle

Let $n < \infty$ be the number of points in the support of V. Then the rank of the operator of multiplication by V equals n. Therefore we have the following result.

Lemma 3.1.

(i) $\operatorname{Spec}_{\text{ess}} H = [0, 8]$.
(ii) *The spectrum of H, outside the interval* $[0, 8]$, *contains at most n eigenvalues on* $(-\infty, 0)$, *and at most n eigenvalues on* $(8, \infty)$.

In the next lemma we develop the results of [2,12] for the 2D case and prove the limiting absorption principle in the sense of operator convergence. It will be needed for the proof of the long-time asymptotics (1.9).

Lemma 3.2. *Let $V \in \mathcal{V}$ and $\sigma > 1/2$. Then the following limit exists as $\varepsilon \to 0+$:*

$$R(\omega \pm i\varepsilon) \xrightarrow{B(\sigma,-\sigma)} R(\omega \pm i0), \quad \omega \in (0, 4) \cup (4, 8). \tag{3.1}$$

Proof. Fix $\omega \in (0, 4) \cup (4, 8)$ and $\sigma > 1/2$. Then Lemma 2.2 yields

$$I + VR_0(\omega \pm i\varepsilon) \xrightarrow{\mathcal{B}(\sigma,\sigma)} I + VR_0(\omega \pm i0), \quad \varepsilon \to 0+;$$

for this, recall that the potential V is assumed to be compactly supported in $\mathbb{Z}^2$. Therefore the convergence $R_0(\omega \pm i\varepsilon) \to R_0(\omega \pm i0)$ in $B(\sigma, -\sigma)$ implies the convergence in $B(\sigma, \sigma)$ after multiplication by V. W. Shaban and B. Vainberg proved in [12] that for $\omega \in (0, 4) \cup (4, 8)$, the operator $I + VR_0(\omega \pm i0)$ has only a trivial kernel. Hence, being Fredholm of index zero, $I + VR_0(\omega \pm i0)$ is invertible, and moreover

$$\big(I + VR_0(\omega \pm i\varepsilon)\big)^{-1} \xrightarrow{\mathcal{B}(\sigma,\sigma)} \big(I + VR_0(\omega \pm i0)\big)^{-1}, \quad \varepsilon \to 0+.$$

Then the representation $R = R_0(I + VR_0)^{-1}$ implies (3.1). □

Remark 3.3. For $\omega \in (0, 4) \cup (0, 8)$ and for any $k \in \mathbb{N}$ we have $\partial_\omega^k R(\omega \pm i0) \in \mathcal{B}(\sigma, -\sigma)$ with $\sigma > 1/2 + k$.

3.2. Asymptotic expansion near elliptic points

Here we obtain an asymptotic expansion for the perturbed resolvent $R(\omega)$ near the elliptic singular points $\omega_1 = 0$ and $\omega_3 = 8$. Fix a finite subset $M \subset \mathbb{Z}^2$ containing $|M|$ points, and denote by $\mathcal{V}_M$ the set of all real-valued potentials supported by M.

Definition 3.4.

(1) A subset $W \subset \mathcal{V}_M$ is called generic if its complement is contained in an algebraic submanifold in $\mathbb{R}^{|M|}$.
(2) We say that a property holds for generic potentials $V \in \mathcal{V}_M$, if it holds for all V from a generic subset $W \subset \mathcal{S}_M$.

Theorem 3.5. *Fix a $\sigma > 3$ and a finite subset $M \subset \mathbb{Z}^2$. Then, for generic potentials $V \in \mathcal{V}_M$ the resolvent $R(\omega)$ has the expansion*

$$R(\omega) = R_1^0 + \frac{R_1^1}{a + \log\omega} + \mathcal{O}(\omega \log^2 \omega), \quad |\omega| \to 0, \ \arg\omega \in (0, 2\pi), \tag{3.2}$$

in the norm of $\mathcal{B}(\sigma, -\sigma)$. Here R_1^0, R_1^1 are operators with kernels $R_1^0(x, y)$ and $R_1^1(x, y)$, respectively.

Proof. The resolvent $R(\omega)$ can be written in the form

$$R(\omega) = R_0(\omega)T^{-1}(\omega), \quad \text{where } T(\omega) := I + V R_0(\omega). \tag{3.3}$$

Due to (2.12) we have

$$T(\omega) = I + V A_0 \log\omega + V B_0 + O(\omega\log\omega), \quad |\omega| \to 0, \ \arg\omega \in (0, 2\pi). \tag{3.4}$$

Lemma 3.6. *The operator $T_0 := I + V B_0 : l^2_\sigma \to l^2_\sigma$ is invertible for generic potentials $V \in \mathcal{V}_M$.*

Proof. Denote by $H(M)$ the space of functions on Z^2 supported by M. The operator $V B_0$ is compact since its range is finite-dimensional. Hence, one needs to check only that the kernel of T_0 is zero. Assume that $T_0 u = 0$, i.e. $u + V B_0 u = 0$. Then $u \in H(M)$, so we have to study the restriction $T_0(M)$ of the operator T_0 on $H(M)$. Hence, $u = 0$ if $\det T_0(M) \neq 0$. Obviously, this determinant is a polynomial of the values of V. This polynomial is not equal to zero identically since it does not vanish at $V = 0$. This completes the proof of the lemma. □

Denote $S_0 = A_0 T_0^{-1} \in \mathcal{B}(\sigma, -\sigma)$. Then (3.3), (3.4) imply, by Lemma 3.6, that

$$\begin{aligned} R(\omega) &= R_0(\omega)T_0^{-1}\big(I + V S_0 \log\omega + O(\omega\log\omega)\big)^{-1} \\ &= \big(S_0 \log\omega + B_0 T_0^{-1} + O(\omega\log\omega)\big)\big(I + V S_0 \log\omega + O(\omega\log\omega)\big)^{-1}. \end{aligned} \tag{3.5}$$

Let us construct the operator $(I + V S_0 \log\omega + O(\omega\log\omega))^{-1}$. Denote by $\operatorname{Ker} S_0 \subset l^2_\sigma$ the kernel of operator S_0. The operator S_0 is one-dimensional, and $S_0 \neq 0$ by (2.13). Hence, $\operatorname{Ker} S_0$ is the subspace of l^2_σ of codimension one.

Lemma 3.7. *For generic potentials $V \in \mathcal{V}_M$, we have $V \notin \operatorname{Ker} S_0$.*

Proof. Inclusion $V \in \operatorname{Ker} S_0$ means that

$$A_0(I + V B_0)^{-1} V = 0. \tag{3.6}$$

For $f = (I + V B_0)^{-1} V$ we have $V = f + V B_0 f$, hence $f \in H(M)$. Therefore, (3.6) becomes

$$\sum_{y \in \mathbb{Z}^2} \big((I + V B_0)^{-1} V\big)(y) = \sum_{y \in M} \big((I + V B_0)^{-1} V\big)(y) = 0$$

by (2.13), and the latter equation is a rational equation for values of V. The rational function is not equal to zero identically, which follows from an analytic expansion of $(I + VB_0)^{-1}V$ at $V = 0$. $\square$

Now for generic potentials V we have $l^2_\sigma = \mathcal{V} \oplus \operatorname{Ker} S_0$, where $\mathcal{V}$ is one-dimensional space of functions proportional to V. We represent any function $f \in l^2_\sigma$ as a vector with components $f_1 \in \mathcal{V}$, $f_2 \in \operatorname{Ker} S_0$, and write the operator $I + VS_0 \log\omega$ in the corresponding matrix form. We have $S_0V = \alpha \neq 0$, and

$$I + VS_0 \log\omega = \begin{pmatrix} 1 + \alpha \log\omega & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.7}$$

Hence, for small $|\omega| > 0$

$$\big(I + VS_0 \log\omega + O(\omega\log\omega)\big)^{-1} = \begin{pmatrix} \frac{1}{1+\alpha\log\omega} & 0 \\ 0 & 1 \end{pmatrix} + O(\omega\log\omega).$$

Therefore, in the matrix form, (3.5) becomes

$$R(\omega) = \left[\begin{pmatrix} S_{11} & 0 \\ S_{21} & 0 \end{pmatrix} \log\omega + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} + O(\omega\log\omega)\right]\left[\begin{pmatrix} \frac{1}{1+\alpha\log\omega} & 0 \\ 0 & 1 \end{pmatrix} + O(\omega\log\omega)\right],$$

where S_{ij} and B_{ij} are the matrix elements of operators S_0 and $B_0T_0^{-1}$, respectively. Then (3.2) follows. This completes the proof of Theorem 3.5. $\square$

Remark 3.8. Expansion of type (3.2) also holds for $R(8 + \omega)$ as $|\omega| \to 0$, $\arg\omega \in (-\pi, \pi)$.

3.3. Asymptotic expansion near hyperbolic points

Now we obtain an asymptotic expansion for the perturbed resolvent $R(\omega)$ near hyperbolic singular point $\omega = 4$.

Theorem 3.9. *Fix a $\sigma > 3$ and a finite subset $M \subset \mathbb{Z}^2$. Then for generic potentials $V \in \mathcal{V}_M$ the following expansion holds*:

$$R(4 + \omega) = R_2^0 + \frac{R_2^1 \log\omega + R_2^2}{\log^2\omega + b\log\omega + c} + \mathcal{O}\big(\omega\log^2\omega\big), \quad |\omega| \to 0,\ \operatorname{Im}\omega > 0, \tag{3.8}$$

in the norm of $B(\sigma, -\sigma)$, R_2^k are some operators with kernels $R_2^k(x, y)$.

Proof. Similar to (3.3), (3.4), we obtain

$$R(4 + \omega) = R_0(4 + \omega)T^{-1}(4 + \omega), \quad \text{where } T(4 + \omega) := I + VR_0(4 + \omega). \tag{3.9}$$

Due to (2.25) we have

$$T(4 + \omega) = I + VD_0 \log\omega + VE_0 + O(\omega\log\omega), \quad |\omega| \to 0,\ \operatorname{Im}\omega > 0. \tag{3.10}$$

Everywhere below we assume that operator $K_0 := I + VE_0 \in \mathcal{B}(\sigma, -\sigma)$ is invertible since this is true for generic potentials V. The proof is similar to Lemma 3.6. Denote $H_0 = D_0 K_0^{-1} \in \mathcal{B}(\sigma, -\sigma)$. Then (3.9), (3.10) imply that

$$\begin{aligned} R(4+\omega) &= R_0(4+\omega)K_0^{-1}\big(I + VH_0\log\omega + O(\omega\log\omega)\big)^{-1} \\ &= \big(H_0\log\omega + D_0K_0^{-1} + O(\omega\log\omega)\big)\big(I + VH_0\log\omega + O(\omega\log\omega)\big)^{-1}. \end{aligned} \tag{3.11}$$

Since $D_0(x-y) = -\frac{i}{4\pi}[(-1)^{(x_1-y_1)} + (-1)^{(x_2-y_2)}]$ (see Proposition 2.6), we have

$$H_0 = D_0(1+VE_0)^{-1} = (-1)^{x_1}H_1 + (-1)^{x_2}H_2, \tag{3.12}$$

where H_k are operators which map any f to a constant

$$H_k f = -\frac{i}{4\pi}\sum_{y\in\mathbb{Z}^2}(-1)^{y_k}\big((I+VE_0)^{-1}f\big)(y).$$

Denote $V_1(y) = (-1)^{y_1}V(y)$ and $V_2(y) = (-1)^{y_2}V(y)$.

Step (i). First we consider the case when V_1 and V_2 are linearly dependent. Denote

$$\mathbb{Z}^2_{\mathrm{even}} = \big\{(y_1,y_2)\in\mathbb{Z}^2\colon\ y_1+y_2 \text{ is even}\big\}, \qquad \mathbb{Z}^2_{\mathrm{odd}} = \big\{(y_1,y_2)\in\mathbb{Z}^2\colon\ y_1+y_2 \text{ is odd}\big\}.$$

Then either $\operatorname{supp} V \subset \mathbb{Z}^2_{\mathrm{even}}$ or $\operatorname{supp} V \subset \mathbb{Z}^2_{\mathrm{odd}}$, and $V_1 = \pm V_2$, respectively.

Note that $\operatorname{supp}(I+VE_0)^{-1}V_1 \in M$, therefore $H_2V_1 = \pm H_1V_1$. Using the same argument as in Lemma 3.7 one can easily show that $V_1 \notin \operatorname{Ker} H_1$ for generic potentials $V \in \mathcal{V}_M$, i.e. $a_1 = H_1V_1 \neq 0$. Decomposition (3.12) implies that

$$VH_0 = V_1H_1 + V_2H_2 = V_1(H_1 \pm H_2),$$

and then $VH_0V_1 = 2V_1a_1$. Hence, VH_0 is a one-dimensional operator with range $\mathcal{V}_1 = \operatorname{span}\{V_1\}$. Therefore, for generic potentials V we have $l^2_\sigma = \mathcal{V}_1 \oplus \operatorname{Ker}(VH_0)$, and in the same way as in Section 3.2, we obtain

$$\begin{aligned} R(4+\omega) = {}&\left[\begin{pmatrix} H^{\pm}_{11} & 0 \\ H^{\pm}_{21} & 0 \end{pmatrix}\log\omega + \begin{pmatrix} E^{\pm}_{11} & E^{\pm}_{12} \\ E^{\pm}_{21} & E^{\pm}_{22} \end{pmatrix} + O(\omega\log\omega)\right] \\ &\times\left[\begin{pmatrix} \frac{1}{1+2a_1\log\omega} & 0 \\ 0 & 1 \end{pmatrix} + O(\omega\log\omega)\right], \end{aligned}$$

where $H^{\pm}_{ij}$ and $E^{\pm}_{ij}$ are matrix elements of operators H_0 and $E_0K_0^{-1}$, respectively. This implies the statement of the theorem in the case of linearly dependent V_1, V_2.

Step (ii). Now we consider the case when V_1 and V_2 are not proportional. In this case the image of VH_0 belongs to $\mathcal{V}_{12} = \operatorname{span}\{V_1, V_2\}$.

Lemma 3.10. *For generic potentials $V \in \mathcal{V}_M$, the operator VH_0 is two-dimensional with the range $\mathcal{V}_{12}$.*

Proof. The operator $VH_0 : \mathcal{V}_{12} \mapsto \mathcal{V}_{12}$ is degenerate if and only if

$$H_1V_1 \cdot H_2V_2 - H_1V_2 \cdot H_2V_1 = 0$$

or, in other notation

$$\sum_{y\in M}(-1)^{y_1}\big((I+VE_0)^{-1}V_1\big)(y)\sum_{y\in M}(-1)^{y_2}\big((I+VE_0)^{-1}V_2\big)(y)$$
$$-\sum_{y\in M}(-1)^{y_1}\big((I+VE_0)^{-1}V_2\big)(y)\sum_{y\in M}(-1)^{y_2}\big((I+VE_0)^{-1}V_1\big)(y)=0,$$

and the latter equation is a rational equation for values of V. The rational function is not equal to zero identically. Indeed, let us consider an analytic expansion of the rational function at $V = 0$. The expansion begins from the second order term which is

$$r_2(V)=\left(\sum_{y\in M}V(y)-\sum_{y\in M}(-1)^{y_1+y_2}V(y)\right)\left(\sum_{y\in M}V(y)+\sum_{y\in M}(-1)^{y_1+y_2}V(y)\right).$$

Since M does not belong to $\mathbb{Z}^2_{\mathrm{even}}$ or $\mathbb{Z}^2_{\mathrm{odd}}$ entirely, then $r_2(V) \not\equiv 0$ on $\mathcal{V}_M$. Hence, for generic potential $V \in \mathcal{V}_M$ the rational function not equal zero. □

Now for generic potentials V we have $l^2_\sigma = \mathcal{V}_{12} \oplus \mathrm{Ker}(VH_0)$. Denote $H_kV_j = a_{kj}$. Then, the operator $I + VH_0 \log\omega = I + V_1H_1 \log\omega + V_2H_2 \log\omega$ in the corresponding matrix form reads:

$$I+VH_0\log\omega=\begin{pmatrix}1+a_{11}\log\omega & a_{12}\log\omega & 0\\ a_{21}\log\omega & 1+a_{22}\log\omega & 0\\ 0&0&1\end{pmatrix}.$$

Hence, for small $|\omega| > 0$

$$\big(I+VH_0\log\omega+O(\omega\log\omega)\big)^{-1}=\begin{pmatrix}\frac{1+a_{22}\log\omega}{\Delta} & \frac{-a_{12}\log\omega}{\Delta} & 0\\ \frac{-a_{21}\log\omega}{\Delta} & \frac{1+a_{11}\log\omega}{\Delta} & 0\\ 0&0&1\end{pmatrix}+O(\omega\log\omega),$$

where $\Delta = 1 + (a_{11}a_{22} - a_{12}a_{21})\log^2\omega + (a_{11} + a_{22})\log\omega$, and $a_{11}a_{22} - a_{12}a_{21} \neq 0$ by Lemma 3.10. Hence,

$$R(4+\omega)=\left[\begin{pmatrix}H_{11}&H_{12}&0\\H_{21}&H_{22}&0\\H_{31}&H_{32}&0\end{pmatrix}\log\omega+\begin{pmatrix}E_{11}&E_{12}&E_{13}\\E_{21}&E_{22}&E_{23}\\E_{31}&E_{32}&E_{33}\end{pmatrix}+O(\omega\log\omega)\right]$$
$$\times\left[\begin{pmatrix}\frac{1+a_{22}\log\omega}{\Delta} & \frac{-a_{12}\log\omega}{\Delta} & 0\\ \frac{-a_{21}\log\omega}{\Delta} & \frac{1+a_{11}\log\omega}{\Delta} & 0\\ 0&0&1\end{pmatrix}+O(\omega\log\omega)\right]$$

where H_{ij} and E_{ij} are matrix elements of operators H_0 and $E_0K_0^{-1}$, respectively. This completes the proof of Theorem 3.9. □

Remark 3.11. Expansions (3.2), (3.8) imply that the resolvent $R(\omega)$ is bounded in a neighborhood of the points ω_k for generic potentials V, though the expansions of unperturbed resolvent (2.12), (2.25) contain the terms $\sim \log\omega$ growing as $\omega \to 0$. Moreover, (3.2), (3.8) imply that

$$R(\omega_k + \omega) = R_k^0 + R_k^1 \log^{-1}\omega + \mathcal{O}\bigl(\log^{-2}\omega\bigr), \quad |\omega| \to 0, \tag{3.13}$$

in the space $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > 3$.

Remark 3.12. The expansions (3.13) in $B(\sigma, -\sigma)$ with $\sigma > 3$ can be differentiated two times in ω. More precisely, $\partial_\omega^2 R(\omega_k + \omega)$ is equal to second derivative of the expansion up to an error $\mathcal{O}(\omega^{-1}\log\omega)$.

Proof. The latter is seen from the formula

$$R(\omega) = \bigl(I + R_0(\omega)V\bigr)^{-1} R_0(\omega),$$

in which $R_0(\omega)$ admits a differentiable asymptotic expansion by Remark 2.8. □

4. Long-time asymptotics

Theorem 4.1. *Let $\sigma > 3$. Then for generic potentials $V \in \mathcal{V}$ the asymptotics* (1.9) *hold, i.e.,*

$$\Bigl\| e^{-itH} - \sum_{j=1}^{n} e^{-it\mu_j} P_j \Bigr\|_{B(\sigma,-\sigma)} = \mathcal{O}\bigl(t^{-1}\log^{-2} t\bigr), \quad t \to \infty. \tag{4.1}$$

Here P_j denote the projections on the eigenspaces corresponding to the eigenvalues $\mu_j \in \mathbb{R} \setminus [0, 8]$, $j = 1, \dots, n$.

Proof.

Step (i). The estimate (4.1) is based on the formula

$$e^{-itH} = -\frac{1}{2\pi i} \oint\limits_{|\omega|=C} e^{-it\omega} R(\omega)\, d\omega, \quad C > \max\bigl\{8; |\mu_j|, j = 1, \dots, n\bigr\}. \tag{4.2}$$

The integral above is equal to the sum of residues at poles of $R(\omega)$ and the integral over the contour around the segment [0, 8], i.e.

$$e^{-itH} - \sum_{j=1}^{n} e^{-it\mu_j} P_j = \frac{1}{2\pi i} \int\limits_{[0,8]} e^{-it\omega}\bigl(R(\omega + i0) - R(\omega - i0)\bigr)\, d\omega.$$

To prove the desired decay for large t, it is convenient to represent the indicator-function χ of interval [0, 8] as

$$\chi(\omega) = \zeta_1(\omega) + \zeta_2(\omega) + \zeta_3(\omega),$$

where $\zeta_i(\omega) \in C^\infty([0,8])$, $\operatorname{supp}\zeta_1 \subset [0, 4-\delta]$, $\operatorname{supp}\zeta_2 \subset [4-2\delta, 4+2\delta]$, $\operatorname{supp}\zeta_3 \subset [4+\delta, 8]$, where $\delta > 0$ is sufficiently small. Then

$$e^{-itH} - \sum_{j=1}^{n} e^{-it\omega_j} P_j = \int\limits_{[0,4-\delta]} e^{-it\omega}\zeta_1(\omega) P(\omega)\, d\omega + \int\limits_{[4-2\delta,4+2\delta]} e^{-it\omega}\zeta_2(\omega) P(\omega)\, d\omega$$
$$+ \int\limits_{[4+\delta,8]} e^{-it\omega}\zeta_3(\omega) P(\omega)\, d\omega = I_1 + I_2 + I_3, \tag{4.3}$$

where $P(\omega) = \frac{1}{2\pi i}(R(\omega + i0) - R(\omega - i0))$.

Step (ii). Let us consider the first summand in the right-hand side of (4.3). The asymptotic expansion for $P(\omega)$ at $\omega = 0$ can be deduced from (3.13):

$$P(\omega) = \frac{2\pi i R_1^1}{\ln\omega(\ln\omega + 2\pi i)} + \mathcal{O}\big(\ln^{-2}\omega\big) = \mathcal{O}\big(\ln^{-2}\omega\big), \quad \omega \to 0,\ \omega > 0.$$

By Lemma 4.2 below we have

$$\int\limits_{[0,4-\delta]} e^{-it\omega}\zeta_1(\omega) P(\omega)\, d\omega = \mathcal{O}\big(t^{-1}\log^{-2} t\big), \quad t \to \infty,$$

in the norm $B(\sigma, -\sigma)$ with $\sigma > 3$, and we obtain the desired decay for the first summand in the right-hand side of (4.3). The same arguments can be used for the third summand in the right-hand side of (4.3).

Step (iii). Let us consider the second summand in the right-hand side of (4.3):

$$I_2(t) = \frac{e^{-4it}}{\pi i} \int\limits_{-2\delta}^{2\delta} e^{-it\omega}\zeta_2(4+\omega) \operatorname{Im} R(4 + \omega + i0)\, d\omega, \tag{4.4}$$

where, by (3.13)

$$R(4 + \omega + i0) = R_2^0 + R_2^1 \log^{-1}(\omega + i0) + \mathcal{O}\big(\ln^{-2}\omega\big). \tag{4.5}$$

The contribution of the first summand in (4.5) to $I_2(t)$ is $\mathcal{O}(t^{-2})$. It can be proved with help of integration by parts 2 times. The contribution of the second term and the remainder are $\mathcal{O}(t^{-1}\ln^{-2} t)$. For the second term this follows directly from Lemma 4.3 below. To estimate the contribution of the remainder we apply Lemma 4.2 below with $\mathbf{B} = B(\sigma, -\sigma)$, $\sigma > 3$. □

Finally, we prepare two lemmas concerning the Fourier transform. The first lemma is an extension of [6, Lemma 10.2] (see also [14, Lemma 10]), and the second one is a corollary of Lemma 9 in [14].

Lemma 4.2. *Assume $\mathcal{B}$ be a Banach space, $d > 0$, and $F \in C(0, d; \mathbf{B})$ satisfies $F(0) = 0$ and $F(\omega) = 0$ for $\omega > d > 0$, $F'' \in L^1(\delta, d; \mathbf{B})$ for any $\delta > 0$. Moreover, $F'(\omega) = \mathcal{O}(\omega^{-1}\ln^{-3}\omega)$ as well as $F''(\omega) = \mathcal{O}(\omega^{-2}\log^{-3}\omega)$ as $\omega \to +0$. Then*

$$\int_0^\infty e^{-it\omega} F(\omega)\, d\omega = \mathcal{O}\big(t^{-1}\ln^{-2} t\big), \quad t \to \infty.$$

Proof. Extending F by $F(\omega) = 0$ for $\omega < 0$, we obtain a function F on $(-\infty, \infty)$ with $F' \in L^1(-\infty, \infty; \mathbf{B})$. For $t > 0$ we have

$$\int_{-\infty}^{\infty} F'(\omega) e^{-it\omega}\, d\omega = -\frac{1}{2}\int_{-\infty}^{\infty}\left(F'\left(\omega + \frac{\pi}{t}\right) - F'(\omega)\right) e^{-it\omega}\, d\omega. \tag{4.6}$$

Finally,

$$\begin{aligned}\int_{-\infty}^{\infty}\left\| F'\left(\omega + \frac{\pi}{t}\right) - F'(\omega)\right\| d\omega &= \int_{-\infty}^{\pi/t} \ldots + \int_{\pi/t}^{\infty} \ldots \leqslant 2\int_0^{2\pi/t} \big\|F'(\omega)\big\|\, d\omega \\ &\quad + \int_{\pi/t}^{\infty} d\omega \int_{\omega}^{\omega+\pi/t} \big\|F''(\mu)\big\|\, d\mu \\ &= \mathcal{O}\big(\ln^{-2} t\big) + \frac{\pi}{t}\int_{\pi/t}^{\infty} \big\|F''(\mu)\big\|\, d\mu = \mathcal{O}\big(\ln^{-2} t\big).\end{aligned} \tag{4.7}$$

Therefore, (4.6) implies that the Fourier transform of F' is $\mathcal{O}(\ln^{-2} t)$, and hence the Fourier transform of F is $\mathcal{O}(t^{-1}\ln^{-2} t)$ as $t \to \infty$. $\square$

Lemma 4.3. *(See [13, Lemma 9].) For any $\zeta(\omega) \in C_0^\infty(\mathbb{R})$ with support in $(-1, 1)$, the following bound holds*:

$$\int e^{-i\omega t}\zeta(\omega)\log^{-1}(\omega + i0)\, d\omega = \mathcal{O}\big(t^{-1}\ln^{-2} t\big), \quad t \to \infty. \tag{4.8}$$

Remark 4.4. Let us stress that the proofs in this section demonstrate that the expansions (3.13) with $k = 1, 2, 3$, provide the long-time asymptotics (4.1).

5. The Klein–Gordon equation

Now we extend the results of Sections 3–4 to the case of the Klein–Gordon equations (1.10), (1.11). Applying the Fourier–Laplace transform

$$\tilde{\boldsymbol{\Psi}}(x, \omega) = \int_0^\infty e^{i\omega t}\boldsymbol{\Psi}(x, t)\, dt, \quad \operatorname{Im}\omega > \alpha_1 > 0,$$

we get the stationary equation

$$(\mathbf{H}-\omega)\tilde{\boldsymbol{\Psi}}(\omega)=-i\boldsymbol{\Psi}_0,\quad \operatorname{Im}\omega>\alpha_1.$$

Let us first consider the resolvent $\mathbf{R}(\omega)=(\mathbf{H}-\omega)^{-1}$ of the operator $\mathbf{H}$.

Lemma 5.1. *If $\omega^2-m^2\in\mathbb{C}\setminus[0,8]$, then the resolvent $\mathbf{R}(\omega)$ can be expressed in terms of the resolvent $R(\omega)$ from* (1.5) *as*

$$\mathbf{R}(\omega)=\begin{pmatrix}\omega R(\omega^2-m^2) & iR(\omega^2-m^2)\\ -i(1+\omega^2R(\omega^2-m^2)) & \omega R(\omega^2-m^2)\end{pmatrix}. \tag{5.1}$$

Proof. The expression for the resolvent $\mathbf{R}_0(\omega)=(\mathbf{H}_0-\omega)^{-1}$ of the free equation with $V=0$ in the case where $\omega^2-m^2\in\mathbb{C}\setminus[0,8]$ can be obtained by the inverse Fourier transform $F^{-1}_{\theta\to x-y}$ of the matrix

$$\frac{1}{\phi(\theta)-(\omega^2-m^2)}\begin{pmatrix}\omega & i\\ -i(\phi(\theta)+m^2) & \omega\end{pmatrix}.$$

Using that by (2.2)

$$F^{-1}_{\theta\to x-y}\left(\frac{1}{\phi(\theta)-(\omega^2-m^2)}\right)=R_0\big(\omega^2-m^2,x,y\big),$$

we get

$$\mathbf{R}_0(\omega)=\begin{pmatrix}\omega R_0(\omega^2-m^2) & iR_0(\omega^2-m^2)\\ -i(1+\omega^2R_0(\omega^2-m^2)) & \omega R_0(\omega^2-m^2)\end{pmatrix}.$$

Put

$$\mathbf{V}=\begin{pmatrix}0 & 0\\ V & 0\end{pmatrix}.$$

Then the formula

$$\mathbf{R}(\omega)=\big(\mathbf{I}-i\mathbf{R}_0(\omega)\mathbf{V}\big)^{-1}\mathbf{R}_0(\omega)$$

yields (5.1). □

The representation (5.1) implies the following properties of the operator $\mathbf{H}$.

(1) By Lemma 3.1 we have that

$$\operatorname{Spec}\mathbf{H}=\big[-\sqrt{m^2+8},-m\big]\cup\big[m,\sqrt{m^2+8}\big].$$

The discrete spectrum of $\mathbf{H}$ is $\nu_j^{\pm}=\pm\sqrt{m^2+\mu_j}$, where μ_j are the eigenvalues of the operator H.

(2) Let $\sigma > 1/2$. By Lemma 3.2, the following limits exist as $\varepsilon \to 0+$.

$$\mathbf{R}(\omega \pm i\varepsilon) \xrightarrow{\mathbf{B}(\sigma,-\sigma)} \mathbf{R}(\omega \pm i0),$$

for $\omega \in (-\sqrt{m^2+8}, -m) \cup (m, \sqrt{m^2+8}) \setminus \{\pm\sqrt{m^2+4}\}$.

(3) Let $\sigma > 3$. For generic potentials V the resolvent $\mathbf{R}$ has an asymptotic expansion at the singular points $\mu = \pm\sqrt{m^2+\omega_k}$, $k = 1, 2, 3$, similar to (3.13):

$$\mathbf{R}(\mu+\omega) = \mathbf{R}^0(\mu) + \mathbf{R}^1(\mu)\log^{-1}\omega + \mathcal{O}\big(\log^{-2}\omega\big), \quad |\omega| \to 0$$

in $\mathbf{B}(\sigma, -\sigma)$.

(4) Let $\sigma > 3$. Similar to Theorem 4.1, for generic potentials V the following asymptotics hold:

$$\left\| e^{-it\mathbf{H}} - \sum_{\pm}\sum_{j=1}^{n} e^{-it\nu_j^{\pm}}\mathbf{P}_j^{\pm} \right\|_{\mathbf{B}(\sigma,-\sigma)} = \mathcal{O}\big(t^{-1}\ln t^{-2}\big), \quad t \to \infty$$

in $\mathbf{B}(\sigma, -\sigma)$. Here $\mathbf{P}_j^{\pm}$ are the projections onto the eigenspaces corresponding to the eigenvalues $\nu_j^{\pm}$, $j = 1, \dots, n$.

Appendix A

Here we prove Lemma 2.5. Let us split the integral as

$$\int_0^{\delta} \frac{F_N(\rho, z)\,d\rho}{\rho-\omega} = \int_0^{\delta} \frac{(F_N(\rho, z) - F_N(|\omega|, z))\,d\rho}{\rho-\omega} + \int_0^{\delta} \frac{F_N(|\omega|, z)\,d\rho}{\rho-\omega} = J_1 + J_2.$$

First we estimate J_2. By (2.21)

$$|J_2| \leqslant \big|F_N\big(|\omega|, z\big)\big| \left| \int_0^{\delta} \frac{d\rho}{\rho-\omega} \right| \leqslant C|z|^{2N}\big|\ln|\omega|\big|.$$

Further we split $J_1 = J_{11} + J_{12}$, where J_1 is integral over $I_1 = (0, \delta) \cap \{|\rho - |\omega|| \leqslant 1/|z|^2\}$, and J_2 is integral over $I_2 = (0, \delta) \setminus I_1$. By (2.21)

$$|J_{11}| \leqslant C|z|^{2N+2} \int_{I_1} \frac{|\rho - |\omega||\,d\rho}{|\rho-\omega|} \leqslant C|z|^{2N+2}\frac{2}{|z|^2} \leqslant C|z|^{2N}$$

since $|\rho - \omega| \geqslant |\rho - |\omega||$, and $|I_1| \leqslant 2/|z|^2$. Finally,

$$|J_{12}| \leqslant C|z|^{2N} \int_{I_2} \frac{d\rho}{|\rho - |\omega||} \leqslant C|z|^{2N}\ln|z|.$$

Appendix B

Here we prove Lemma 2.7. We estimate the integral only over $\Pi_+ = \{0 \leqslant \rho_1, \rho_2 \leqslant \delta\}$. The integral over $\Pi \setminus \Pi_+$ can been estimated similarly. Let us split the integral over Π_+ as

$$\int\limits_{\Pi_+} \frac{H_N(r,\rho_1,z)\,d\rho_1\,d\rho_2}{(\rho_1-\omega)r} = \int\limits_{\Pi_+} \frac{(H_N(r,\rho_1,z)-H_N(r,|\omega|,z))\,d\rho_1\,d\rho_2}{(\rho_1-\omega)r} + \int\limits_{\Pi_+} \frac{H_N(r,|\omega|,z)\,d\rho_1\,d\rho_2}{(\rho_1-\omega)r} = J_1 + J_2.$$

Similar to (2.34) we obtain

$$|J_2| = \left| \int\limits_0^\delta H_N\bigl(r,|\omega|,z\bigr)\,dr \int\limits_0^\pi \frac{d\psi}{r\cos\psi-\omega} \right| = \pi \left| \int\limits_0^\delta \frac{H_N(r,|\omega|,z)\,dr}{\sqrt{r^2-\omega^2}} \right| \leqslant C|z|^{2N}\bigl|\ln|\omega|\bigr|.$$

Further, we split J_1 as

$$J_1 = J_{11} + J_{12} + J_{13},$$

where J_{11} is the integral over $\Pi_1 = \{(\rho_1,\rho_2)\in\Pi_+\colon |r| < 1/|z|^2\}$, J_{12} is the integral over $\Pi_2 = \{(\rho_1,\rho_2)\in\Pi_+\setminus\Pi_1\colon |\rho_1-|\omega|| < 1/|z|^4\}$, and J_{13} is the integral over $\Pi_3 = \Pi_+\setminus(\Pi_1\cup\Pi_2)$ (see Fig. 1). By (2.32) we obtain

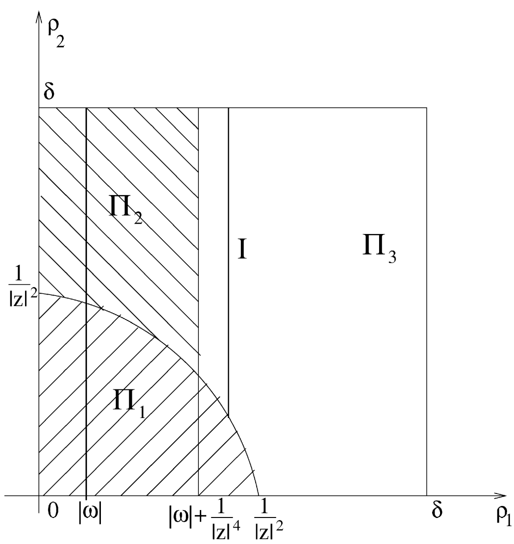

Fig. 1. The case $|\omega| - 1/|z|^4 < 0$.

$$|J_{11}| \leqslant C|z|^{2N+2} \int_{\Pi_1} \frac{|\rho_1 - |\omega||\, d\rho_1\, d\rho_2}{|\rho_1 - \omega| r} \leqslant C|z|^{2N+2} \int_0^{\pi} d\psi \int_0^{1/|z|^2} dr \leqslant C|z|^{2N},$$

since $|\rho_1 - \omega| \geqslant |\rho_1 - |\omega||$. Next,

$$|J_{12}| \leqslant C|z|^{2N+2} \int_{\Pi_2} \frac{|\rho_1 - |\omega||\, d\rho_1\, d\rho_2}{|\rho_1 - \omega| r} \leqslant C|z|^{2N+2}|z|^2 \frac{\delta}{|z|^4} \leqslant C|z|^{2N},$$

since $r \geqslant 1/|z|^2$, and $|\Pi_2| \leqslant 2\delta/|z|^4$. Finally, by (2.32) we have

$$|J_{13}| \leqslant C|z|^{2N} \int_{\Pi_3} \frac{d\rho_1\, d\rho_2}{|\rho_1 - |\omega||\sqrt{\rho_1^2 + \rho_2^2}} \leqslant C|z|^{2N} \ln^2 |z|,$$

since for any vertical interval $I \in \Pi_3$ (see Fig. 1)

$$\int_I \frac{d\rho_2}{\sqrt{\rho_1^2 + \rho_2^2}} = \ln\bigl(\rho_2 + \sqrt{\rho_1^2 + \rho_2^2}\bigr) \leqslant C \ln |z|.$$

Appendix C

Here we derive the asymptotics (3.13) under the condition of boundedness of the matrix elements of the resolvent $R(\omega, x, y)$ near the singular points $\omega = 0, 4, 8$. We shall apply methods [3] relying on the asymptotic expansions of the free resolvent obtained in Section 2. For example, we consider the case $\omega = 0$. Let us recall that $V \in \mathcal{V}$.

Theorem C.1. *Let the matrix elements $R(\omega, x, y)$ of the resolvent $R(\omega)$ be bounded near $\omega = 0$, i.e.*

$$\bigl|R(\omega, x, y)\bigr| \leqslant C(x, y), \quad |\omega| \leqslant \varepsilon,\ 0 < \arg\omega < 2\pi. \tag{C.1}$$

Then the following expansion holds in the space $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > 3$:

$$R(\omega) = R^0 + R^1 \log^{-1} \omega + \mathcal{O}\bigl(\log^{-2} \omega\bigr), \quad |\omega| \to 0,\ 0 < \arg\omega < 2\pi. \tag{C.2}$$

Proof. The resolvent $R(\omega)$ can be written in the form

$$R(\omega) = R_0(\omega) T^{-1}(\omega), \quad \text{where } T(\omega) := I + V R_0(\omega). \tag{C.3}$$

Let M be a finite subset of Z^2 with $|M|$ points which contains the support of the potential V. Denote by l^2_M the space of elements from $l^2(Z^2)$ supported on M. Let $T_M(\omega): l^2_M \to l^2_M$ be the restriction of the operator $T(\omega)$ onto l^2_M. Using the Kramer rule, as in the proof of [3, Theorem 8], we obtain the asymptotic expansion

$$T_M^{-1}(\omega) = \omega^\alpha \log^\beta \omega\bigl[T_0 + T_1 \log^{-1} \omega + T_2 \log^{-2} \omega + O\bigl(\log^{-3} \omega\bigr)\bigr], \quad \omega \to 0,$$

with some matrices $T_0, T_1, T_2 : l^2_M \to l^2_M$, where $T_0 \neq 0$, and integers α and β. Taking into account (2.12) and

$$T^{-1}(\omega) = I - T^{-1}(\omega) V R_0(\omega) = I - T_M^{-1}(\omega) V R_0(\omega),$$

we get the following expansion in $\mathcal{B}(\sigma, \sigma)$

$$T^{-1}(\omega) = \omega^\alpha \log^\gamma \omega \big[\hat{T}_0 + \hat{T}_1 \log^{-1} \omega + \hat{T}_2 \log^{-2} \omega + O\big(\log^{-3} \omega\big)\big], \quad \omega \to 0, \tag{C.4}$$

with some $\hat{T}_k \in \mathcal{B}(\sigma, \sigma)$ instead of T_k and $\hat{T}_0 \neq 0$. Substituting (2.12) and (C.4) into (C.3), we obtain the following expansion

$$R(\omega) = \omega^\alpha \log^{\gamma+1} \omega \big[A_0 \hat{T}_0 + (A_0 \hat{T}_1 + B_0 \hat{T}_0) \log^{-1} \omega + O\big(\log^{-2} \omega\big)\big], \quad \omega \to 0, \tag{C.5}$$

in the space $\mathcal{B}(\sigma, \sigma)$ with $\sigma > 3$. Here $A_0 \hat{T}_1 f$ is a constant function, and $B_0 \hat{T}_0 f$ satisfies the equation $H_0 B_0 \hat{T}_0 f = \hat{T}_0 f$ since $H_0 B_0 = 1$. Hence $H_0 u = \hat{T}_0 f$ for $u = (A_0 \hat{T}_1 + B_0 \hat{T}_0) f$. Since $\hat{T}_0 \neq 0$, there exists $f \in l^2_\sigma$ for which the second term in the right-hand side of (C.5) does not vanish. Hence, the expansion (C.5) has the form

$$R(\omega) = \omega^\alpha \log^\nu \omega \big[R^0 + R^1 \log^{-1} \omega + \mathcal{O}\big(\log^{-2} \omega\big)\big], \quad |\omega| \to 0, \tag{C.6}$$

where $R^0 \neq 0$ and $\nu = \gamma + 1$ if $A_0 \hat{T}_0 \neq 0$, or $\nu = \gamma$ if $A_0 \hat{T}_0 = 0$. Now from the boundedness of the resolvent it follows that $|\omega^\alpha \log^\nu \omega| \leqslant C$, $\omega \to 0$. On the other hand, $\omega^\alpha \log^\nu \omega$ cannot vanish at $\omega = 0$, since $HR(0) f = f$. Thus, $\alpha = \nu = 0$, and (C.6) coincides with (C.2). $\square$

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